

# ON STICKELBERGER ELEMENTS FOR $\mathbb{Q}(\zeta_{p^{n+1}})^+$ AND $p$ -ADIC $L$ -FUNCTIONS

TIMOTHY ALL

*For my parents*

ABSTRACT. We give a survey of a couple known constructions of  $p$ -adic  $L$ -functions including Iwasawa's construction from classical Stickelberger elements. We then construct "real" Stickelberger elements, i.e., explicit elements in the Galois group ring with  $\mathbb{Z}_p$  coefficients that annihilate the Sylow  $p$ -subgroup of the ideal class group of  $\mathbb{Q}(\zeta_{p^{n+1}})^+$ . In analogy with Iwasawa's work, we show that these elements are coherent in  $\mathbb{Z}_p$ -towers and give rise to twisted  $p$ -adic  $L$ -functions.

## 1. INTRODUCTION

Let  $\chi$  be a Dirichlet character. The  $L$ -function attached to  $\chi$  is defined on  $\Re(s) > 1$  by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

It would be an understatement to say that these functions play an important role in algebraic number theory. To illustrate what we mean, let  $k$  be an abelian number field with Galois group  $G_k$ . We write  $\widehat{G}_k = \text{Hom}_{\mathbb{Z}}(G_k, \mathbb{C})$ . By the Kronecker-Weber theorem,  $k \subset \mathbb{Q}(\zeta_m)$  for some minimal positive integer  $m$ , so we may associate  $\widehat{G}_k$  with a certain subgroup of Dirichlet characters on  $(\mathbb{Z}/m\mathbb{Z})^\times$ . Let  $h_k$  denote the class number of  $k$ ,  $R(k)$  the regulator of  $k$ ,  $w_k$  the order of the group of roots of unity of  $k$ , and  $d_k$  the discriminant of  $k$ . We have the following classical result:

$$(1.1) \quad \frac{2^{r_1} (2\pi)^{r_2} h_k R(k)}{w_k \sqrt{|d_k|}} = \prod_{\substack{\chi \in \widehat{G}_k \\ \chi \neq 1}} L(1, \chi),$$

where  $r_1$  (resp.  $r_2$ ) is the number of real places (resp. pairs of complex places) of  $k$ . The above formula is a first indication that the  $L$ -functions attached to characters in  $\widehat{G}_k$  encode fundamental algebraic properties of  $k$ .

Of particular interest is the class number  $h_k$ . This is because many Diophantine equations can be attacked by converting to an ideal equation in

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2010 *Mathematics Subject Classification.* 11R23.

*Key words and phrases.* Stickelberger elements,  $p$ -adic  $L$ -functions, distributions, class group, real abelian number fields.

the ring of integers of an appropriate number field. The classical example is the Fermat equation: let  $p$  be an odd prime (from here on out) and consider  $x^p + y^p = z^p$ . Assuming a non-trivial solution exists, we obtain the ideal equation

$$\prod_{j=0}^{p-1} (x + y\zeta_p^j) = (z)^p$$

in  $\mathbb{Z}[\zeta_p]$  where  $\zeta_p = e^{2\pi i/p}$ . Under some additional assumptions, the ideals  $(x + y\zeta_p^j)$  can be shown to be co-prime, thus  $(x + \zeta_p y) = \mathfrak{a}^p$  for some ideal  $\mathfrak{a} \subset \mathbb{Z}[\zeta_p]$ . If  $p \nmid h_k$ , then this implies that  $\mathfrak{a} = (\alpha)$  is principal. On the other hand, a relation of the sort  $x + y\zeta_p = \epsilon\alpha^p$  for a unit  $\epsilon$  is impossible. This type of approach can be applied to many Diophantine equations, and the quality of  $h_k \bmod p$  is often a feature of those arguments.

In order to better understand the quality of  $h_k \bmod p$  or, even better,  $h_k \bmod p^n$ , it makes sense considering Equation (1.1) to search for a  $p$ -adic analogue of  $L(s, \chi)$ . In particular, we want a continuous function  $L_p(\cdot, \chi) : \mathbb{Z}_p \rightarrow \mathbb{C}_p$  such that

$$(1.2) \quad L_p(1 - n, \chi) = (1 - \chi(p)p^{n-1})L(1 - n, \chi)^1$$

for every integer  $n > 0$  satisfying  $n \equiv 1 \bmod p - 1$ . Such a function is necessarily unique. Moreover, if  $\chi$  is odd then  $L_p(s, \chi)$  is identically zero. This is because

$$L(1 - n, \chi) = -B_{n, \chi}/n,$$

for all integers  $n \geq 1$  where the  $B_{n, \chi}$  are the generalized Bernoulli numbers. Specifically, the rational numbers  $B_{n, \chi}$  are defined by the following formula:

$$(1.3) \quad \sum_{a=1}^{f_\chi} \frac{\chi(a)te^{at}}{e^{f_\chi t} - 1} = \sum_{n=0}^{\infty} B_{n, \chi} \frac{t^n}{n!},$$

where  $f_\chi$  is the conductor of  $\chi$ . But  $B_{n, \chi} = 0$  in the event that  $n \equiv 1 \bmod p - 1$  and  $\chi$  is odd. Since  $L_p(1 - n, \chi)$  vanishes on a dense subset of  $\mathbb{Z}_p$ , it follows that  $L_p(s, \chi)$  is identically zero in this case.

So it would seem  $p$ -adic  $L$ -functions are naturally suited to comment on totally real number fields. In fact, we have the following  $p$ -adic analogue of Equation (1.1). For a totally real abelian number field  $k$  of degree  $n$  over  $\mathbb{Q}$ , we have

$$(1.4) \quad \frac{2^{n-1}h_k R_p(k)}{\sqrt{d_k}} = \prod_{\substack{\chi \in \hat{G} \\ \chi \neq 1}} \left(1 - \frac{\chi(p)}{p}\right)^{-1} L_p(1, \chi),$$

where  $R_p(k)$  is the  $p$ -adic Regulator of  $k$ . The quantity  $R_p(k)$  is defined just as  $R(k)$  but with the Iwasawa  $p$ -adic logarithm replacing the usual logarithm. Note that the terms  $R_p(k)$  and  $\sqrt{d_k}$  are only defined up to a sign, but  $R_p(k)/\sqrt{d_k}$  can be defined without ambiguity. It is known that

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<sup>1</sup>The ‘‘Euler factor’’ at  $p$  must be removed for issues relating to convergence.

$R_p(k)$  is non-zero for abelian fields and conjectured to be non-zero for all number fields  $k$ , but a proof has yet to be found.

Various constructions of  $p$ -adic  $L$ -functions have been given over the years, the first of which was offered by Leopoldt and Kubota [6]. Specifically, they show that the function

$$\frac{1}{s-1} \lim_{n \rightarrow \infty} \frac{1}{p^n[f,p]} \sum_{\substack{x=1 \\ (x,p)=1}}^{p^n[f,p]} \chi(x) \sum_{n=0}^{\infty} \binom{1-s}{n} (\langle x \rangle - 1)^n,$$

where  $[f, p]$  is the least common multiple of  $f$  and  $p$ , in fact, satisfies Footnote 1. Iwasawa [4] famously constructed  $L_p(s, \chi)$  from the classical Stickelberger elements in the theory of cyclotomic fields. As a result of his construction, he showed that there exists a power series  $F(T) \in \mathbb{Z}_p(\chi)[[T-1]]$  such that  $F((1+p)^s) = L_p(s, \chi)$  where  $\mathbb{Z}_p(\chi)$  is the ring of integers of  $\mathbb{Q}_p(\chi(1), \chi(2), \dots)$ . This result is not only very deep by the nature of its construction, but it is also very useful.

It is a theme of this article that constructions of  $p$ -adic  $L$ -functions, and other related objects, benefit from a certain measure theoretic point of view. For example, the Leopoldt-Kubota construction revolves around expressing  $L_p(s, \chi)$  as the  $\Gamma$ -transform of a measure arising from a certain rational function related to Equation (1.3). In §2, we outline the notation and setup to appreciate this measure theoretic viewpoint.

In §3, we discuss Iwasawa's construction using odd-parts of the classical Stickelberger elements. In §4, we discuss a new construction that mirrors that of Iwasawa but takes place in the "plus-part". In particular, we construct real analogs of Stickelberger elements, i.e., explicit elements in a Galois group ring with  $\mathbb{Z}_p$ -coefficients that annihilate the Sylow  $p$ -subgroup of the ideal class group of  $\mathbb{Q}(\zeta_{p^{n+1}})^+$ . The Kummer-Vandiver conjecture states that  $p \nmid h_{k_n^+}$ , so these elements are of particular interest. We then show that the even-parts of these new Stickelberger elements give rise to twisted  $p$ -adic  $L$ -functions. In §5, we summarize and compare results across the previous sections. Our main result is Theorem 5.6.

## 2. PRELIMINARIES

Let  $\mathfrak{o}$  be the ring of integers of  $K$ , a finite extension of  $\mathbb{Q}_p$ . Let  $\Gamma$  be a group topologically isomorphic to  $\mathbb{Z}_p$ . We write  $\Gamma_n = \Gamma/\Gamma^{p^n}$ , this gives us  $\Gamma = \varprojlim \Gamma_n$  under the natural maps. Let  $\gamma$  be a fixed topological generator for  $\Gamma$ , and write  $\gamma_n$  for  $\gamma \bmod \Gamma^{p^n}$ . For any commutative ring  $R$ , we set

$$R[[\Gamma]] := \varprojlim R[\Gamma_n],$$

$$R[[T-1]] := \text{the power series ring with coefficients in } R.$$

An  $R$ -valued *distribution* on  $\Gamma$  is a collection of maps  $\{\mu_n : \Gamma_n \rightarrow R\}$  with the following property:

$$\mu_n(x) = \sum_{y \mapsto x} \mu_{n+1}(y).$$

We typically write  $\mu(a + p^n \mathbb{Z}_p)$  in place of  $\mu_n(\gamma_n^a)$ . Let  $R(\Gamma)$  denote the ring (under convolution) of  $R$ -valued distributions. For a continuous function  $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$  and a distribution  $\mu \in \mathbb{C}_p(\Gamma)$ , we write

$$\int_{\mathbb{Z}_p} f \, d\mu := \lim_{n \rightarrow \infty} \sum_{x=0}^{p^n-1} f(x) \mu(x + p^n \mathbb{Z}_p),$$

if it exists. The integral is guaranteed to exist if there exists an integer  $M$  such that  $|\mu(x + p^n \mathbb{Z}_p)|_p < M$  for all  $x$  and  $n$ . In this case, we say  $\mu$  is *bounded*. We write  $F_\mu(T)$  for the *Fourier transform* of  $\mu$ :

$$F_\mu(T) = \int_{\mathbb{Z}_p} T^x \, d\mu(x) = \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p} \binom{x}{n} \, d\mu(x) \right) (T-1)^n \in \mathbb{C}_p[[T-1]].$$

If  $F \in \mathfrak{o}[[T-1]]$ , then there exists a distribution  $\mu_F$  whose Fourier transform is  $F$ , specifically

$$\mu_F(a + p^n \mathbb{Z}_p) = \frac{1}{p^n} \sum_{x=0}^{p^n-1} \zeta_{p^n}^{-ax} F(\zeta_{p^n}^x).$$

Finally, for  $\mu \in R(\Gamma)$ , let  $\theta_\mu$  denote the following element of  $R[\Gamma_n]$ :

$$\theta_\mu = \left( \sum_{a=0}^{p^n-1} \mu(a + p^n \mathbb{Z}_p) \gamma_n^{-a} \right).$$

The map  $R(\Gamma) \rightarrow R[[\Gamma]]$  defined by  $\mu \rightarrow \theta_\mu$  is an isomorphism of rings. Even better, when  $R = \mathfrak{o}$ , the following isomorphisms fit into a commutative diagram:

$$\begin{array}{ccc} \mathfrak{o}(\Gamma) & \longrightarrow & \mathfrak{o}[[T-1]] \\ & \searrow & \downarrow \\ & & \mathfrak{o}[[\Gamma]] \end{array} \quad \text{given by} \quad \begin{array}{ccc} \mu & \longmapsto & F_\mu \\ & \searrow & \downarrow \\ & & \theta_\mu \end{array} \quad \text{or} \quad \begin{array}{ccc} \mu_F & \longmapsto & F \\ & \searrow & \downarrow \\ & & \theta_F \end{array}$$

In this setup,  $\gamma^{-1} \in \mathfrak{o}[[\Gamma]]$  corresponds to the point-mass distribution centered at 1. Accordingly, we have  $\gamma^{-1}$  corresponds to  $T$  in  $\mathfrak{o}[[T-1]]$ , so  $\gamma$  corresponds to  $1/T \in \mathfrak{o}[[T-1]]$ .

Alternatively, we have  $\mathfrak{o}[[\Gamma]] \simeq \mathfrak{o}[[X]]$  induced by the Iwasawa isomorphism

$$\begin{aligned} \mathfrak{o}[\Gamma_n] &\rightarrow \mathfrak{o}[[X]]/(1 - (1+X)^{p^n}) \\ \gamma_n &\mapsto 1 + X \bmod 1 - (1+X)^{p^n}. \end{aligned}$$

If  $\mu$  is the  $\mathfrak{o}$ -valued distribution underlying  $\theta_\mu \in \mathfrak{o}[[\Gamma]]$ , we write  $I_\mu(X) \in \mathfrak{o}[[X]]$  to denote the corresponding power series under the Iwasawa isomorphism.

In this setup,  $\gamma$  corresponds to  $1 + X$ . Altogether, we have  $\mathfrak{o}[[\Gamma]] \simeq \mathfrak{o}[[X]] \simeq \mathfrak{o}[[T - 1]]$  via  $\gamma \mapsto 1 + X \mapsto T$ , so

$$I_\mu(X) = F_\mu\left(\frac{1}{X+1}\right) \quad \text{and} \quad F_\mu(T) = I_\mu\left(\frac{1}{T} - 1\right).$$

Any  $x \in \mathbb{Z}_p^\times$  can be written as a product of a  $p - 1$ -st root of unity  $\omega(x)$  and an element  $\langle x \rangle \in 1 + p\mathbb{Z}_p$ . We naturally think of  $\omega$  as a Dirichlet character of conductor  $p$ . For a distribution  $\mu \in \mathfrak{o}(\Gamma)$ , we define the *Gamma transform* of  $\mu$ , denoted  $G_\mu(s)$ , by

$$G_\mu(s) = \int_{\mathbb{Z}_p^\times} \langle x \rangle^s d\mu(x).$$

As mentioned before, this measure-theoretic language facilitates the types of interpolation problems encountered when constructing  $L_p(s, \chi)$ . For example, let  $\mathfrak{o}$  be the ring of integers of a finite extension of  $\mathbb{Z}_p$  containing the values of the Dirichlet character  $\chi$ . Let  $c > 1$  be an integer co-prime to  $p$ , and let  $\chi_c$  be the function defined by

$$\chi_c(a) = \begin{cases} \chi(a), & c \nmid a \\ \chi(a)(1 - c), & c \mid a. \end{cases}$$

Let  $F(T)$  be the rational function

$$F(T) = \frac{\sum_{a=1}^{f_\chi} \chi_c(a) T^a}{1 - T^{f_\chi}}.$$

One can show that  $F(T) \in \mathfrak{o}[[T - 1]]$  (by virtue of our definition for  $\chi_c$ ), so let  $\mu_F \in \mathfrak{o}(\Gamma)$  be the corresponding distribution. By differentiating the defining formula for  $F$ , we see that for non-negative integers  $k$

$$\int_{\mathbb{Z}_p} x^k d\mu_F(x) = \left( T \frac{d}{dT} \right)^k F(T) \Big|_{T=1}.$$

In particular, this gives us

$$\int_{\mathbb{Z}_p} x^k d\mu_F(x) = \frac{d^k}{dz^k} F(e^z) \Big|_{z=0} = L(-k, \chi_c).$$

The last equality in the above follows from the fact that the coefficient on  $z^k/k!$  in the Laurent expansion of  $F(e^z)$  at  $z = 0$  is equal to  $L(-k, \chi_c)$  where  $L(s, \chi_c)$  is the analytic continuation of the Dirichlet series defined for  $\Re(s) > 1$  by

$$L(s, \chi_c) = \sum_{n=1}^{\infty} \frac{\chi_c(n)}{n^s}.$$

From the above expression, we get that

$$L(-k, \chi_c) = (1 - \chi(c)c^{k+1})L(-k, \chi).$$

The Gamma transform  $G_{\mu_F}(s)$  is an interpolation of the integrals of  $x^k$  with respect to  $\mu_F$ . Taking the limit over integers  $k$  such that  $k \rightarrow s$  ( $p$ -adically) and  $k \equiv 0 \pmod{p-1}$ , we see that

$$\lim \int_{\mathbb{Z}_p} x^k d\mu_F(x) = \int_{\mathbb{Z}_p^\times} \langle x \rangle^s d\mu_F(x).$$

Whence

$$(2.1) \quad \frac{G_{\mu_F}(s)}{1 - \chi(c)\langle c \rangle^{s+1}} = L_p(-s, \chi).$$

The fact that the  $p$ -adic  $L$ -function is the Gamma transform of a rational function was a crucial ingredient in Sinnott's proof [10] of the vanishing of the  $\mu$ -invariant for the cyclotomic  $\mathbb{Z}_p$ -extension of an abelian number field.

### 3. THE MINUS SIDE

From Equation (2.1), one can deduce that there are elements  $G, H \in \mathfrak{o}[[T-1]]$  such that

$$L_p(-s, \chi) = \frac{G(\kappa^s)}{H(\kappa^s)},$$

where  $\kappa$  is a fixed generator for  $1 + p\mathbb{Z}_p$ . The fact that this is true was originally proven by Iwasawa [4], though he proves even more. In particular, if  $\chi$  (resp.  $\psi$ ) is a tame (resp. wild) character, then there exists  $G_\chi, H_\chi \in \mathfrak{o}[[T-1]]$  such that

$$L_p(s, \chi\psi) = \frac{G_\chi(\zeta_\psi \kappa^s)}{H_\chi(\zeta_\psi \kappa^s)}$$

where  $\zeta_\psi = \psi^{-1}(\kappa)$ . This is a very important result for a number of reasons. For example, the right hand side of Equation (1.4) can now be expressed with a handful of power series. This allows one to study the growth of the class number of  $k_n$  and  $k_n^+$  as  $n$  grows (see [5]).

For clarity of exposition, we restrict ourselves to the following special case. Let  $k_n = \mathbb{Q}(\zeta_{p^{n+1}})$ . We have the decomposition

$$W \times U = \mathbb{Z}_p^\times$$

where  $W$  is the set of  $p-1$ -st roots of unity in  $\mathbb{Z}_p$  and  $U = 1 + p\mathbb{Z}_p$ . We write  $\sigma_a$  to denote the automorphism  $\zeta_{p^{n+1}} \mapsto \zeta_{p^{n+1}}^a$  in  $G_n = \text{Gal}(k_n/\mathbb{Q})$ . For any  $x \in \mathbb{Z}_p$ , we write  $s_n(x)$  to denote the unique integer in the interval  $[0, p^{n+1})$  satisfying  $s_n(x) \equiv x \pmod{p^{n+1}}$ . Then  $s_n$  provides an onto morphism

$$\begin{aligned} \mathbb{Z}_p^\times &\rightarrow G_n \\ x &\mapsto \sigma_{s_n(x)}. \end{aligned}$$

Let  $\gamma_n = \sigma_{s_n(\kappa)} \in \text{Gal}(k_n/k_0) = \Gamma_n$ , so we have

$$\zeta_{p^{n+1}}^{\sigma_{s_n(\zeta_{p^{n+1}})} \gamma_n^a} = \zeta_{p^{n+1}}^{\zeta_{p^{n+1}}^a} = \zeta_{p^{n+1}}^{\sigma_{s_n(\zeta_{p^{n+1}})} \gamma_n^a}.$$

**3.1. Stickelberger Elements.** Let  $\ell$  be a rational prime congruent to 1 mod  $p^{n+1}$ . Let  $\tau$  be a generator for  $\text{Gal}(k_n(\zeta_\ell)/k_n)$ , and consider the Gauss sum

$$g_n(\ell) = - \sum_{b=1}^{\ell-1} \zeta_{p^{n+1}}^b \zeta_\ell^{\tau^b}.$$

If  $\lambda$  is a prime of  $k_n$  above  $\ell$ , then for some integer  $c$  with  $(c, p) = 1$ , one shows that  $g_n(\ell)^{c\sigma_c^{-1}-1} \in k_n$  and

$$(g_n(\ell)^{c\sigma_c^{-1}-1}) = \lambda^{(c-\sigma_c)\theta_n},$$

where  $\theta_n$  is the group ring element

$$\frac{1}{p^{n+1}} \sum_{\substack{a=1 \\ (a,p)=1}}^{p^{n+1}} a\sigma_a^{-1} \in \mathbb{Q}[\text{Gal}(k_n/\mathbb{Q})].$$

The factorization of  $g_n(\ell)$  was originally of interest because of problems pertaining to reciprocity, but upon further inspection it's quite peculiar. After all, if we change the prime  $\ell$ , the group ring element  $\theta_n$  still remains! This gestalt shift in viewing the factorization of the Gauss sums  $g_n(\ell)$  leads us to

**Theorem 3.1** (Stickelberger). *For any  $\beta \in \mathbb{Z}[G]$  such that  $\beta\theta_n \in \mathbb{Z}[G]$ , (for instance, let  $c \in \mathbb{Z}$  such that  $(c, p) = 1$  and take  $\beta = (c - \sigma_c)$ ), we have that  $\beta\theta_n$  annihilates the ideal class group of  $k_n$ .*

This result allows us to investigate the class group of  $k_n$  using an element that only depends on the conductor of  $k_n$ . See [9] for more information on Stickelberger elements in the “minus-part”.

**3.2.  $p$ -adic  $L$ -functions.** Let  $\chi$  be a non-trivial character of  $W \simeq \text{Gal}(k_0/\mathbb{Q})$ , and let  $e_\chi$  denote the idempotent

$$\frac{1}{|W|} \sum_{\zeta \in W} \chi(\zeta) \sigma_{s_n(\zeta)}^{-1}.$$

Let  $\theta_n(\chi) \in \mathbb{Q}_p[\Gamma_n]$  denote the  $\chi$ -component of the Stickelberger element  $\theta_n$ :  $e_\chi \theta_n = \theta_n(\chi) e_\chi$ . We have

$$\theta_n(\chi) = \sum_{a=0}^{p^n-1} \left( \frac{-1}{p^{n+1}} \sum_{\zeta \in W} s_n(\zeta \kappa^a) \chi(\zeta)^{-1} \right) \gamma_n^{-a}.$$

**Lemma 3.2** (Iwasawa). *The elements  $\theta_n(\chi)$  are coherent, i.e., the sequence  $(\theta_n(\chi))$  belongs to  $\mathbb{Q}_p[[\Gamma]]$ .*

*Proof.* The elements  $\theta_n(\chi)$  are coherent if and only if the map

$$\mu_\theta^\chi(a + p^n \mathbb{Z}_p) = \frac{-1}{p^{n+1}} \sum_{\zeta \in W} s_n(\zeta \kappa^a) \chi(\zeta)^{-1}$$

forms a distribution. Note that

$$\sum_{i=0}^{p-1} \mu_{\theta}^{\chi}(a + ip^n + p^{n+1}\mathbb{Z}_p) = \frac{-1}{p^{n+2}} \sum_{\zeta \in W} \left( \sum_{i=0}^{p-1} s_{n+1}(\zeta \kappa^{a+ip^n}) \right) \chi(\zeta)^{-1},$$

while

$$s_{n+1}(\kappa^{a+ip^n} \zeta) = s_n(\zeta \kappa^a) + s_0(x_{n+1} + iy_0)$$

where  $x_{n+1}$  (resp.  $y_0$ ) is the  $n+1$ -st (resp. 0-th) coordinate in the  $p$ -adic expansion of  $\zeta \kappa^a$  (resp.  $\zeta$ ). As  $i$  varies between 0 and  $p-1$ , so does  $s_0(x_{n+1} + iy_0)$ , hence

$$\sum_{i=0}^{p-1} s_{n+1}(\zeta \kappa^{a+ip^n}) = \frac{p(p-1)}{2}.$$

Since  $\chi \neq 1$ , it follows that

$$\sum_{i=0}^{p-1} \mu_{\theta}^{\chi}(a + ip^n + p^{n+1}\mathbb{Z}_p) = \mu_{\theta}^{\chi}(a + p^n\mathbb{Z}_p).$$

This completes the proof of the lemma.  $\square$

*Remark 3.3.* If  $\chi \neq 1$  is even, then

$$\mu_{\theta}^{\chi}(a + p^n\mathbb{Z}_p) = \frac{-1}{p^{n+1}} \sum_{\zeta \in W} s_n(-\zeta \kappa^a) \chi(\zeta)^{-1}.$$

But  $s_n(-x) = p^{n+1} - s_n(x)$  which implies  $\mu_{\theta}^{\chi}(a + p^n\mathbb{Z}_p) = -\mu_{\theta}^{\chi}(a + p^n\mathbb{Z}_p)$ . So if  $\chi$  is even, then  $\mu_{\theta}^{\chi}(a + p^n\mathbb{Z}_p) = 0$  for all  $a$ . This shows us that in order to obtain interesting results, we must take  $\chi$  to be an *odd* character. Therefore, we assume  $\chi$  is odd for the remainder of this section.

Now that we have a coherent sequence  $\theta_n(\chi)$  and a matching distribution  $\mu_{\theta}^{\chi}$  we'd like to know what analytic functions come with them. But unfortunately,  $\mu_{\theta}^{\chi}$  is generally an unbounded distribution. On the other hand, this distribution was born out of Stickelberger elements and we know how to *integralize* them. To boot, the *integralizers*  $1 - c\sigma_c^{-1}$  are, in fact, coherent themselves! This is a crucial observation.

Let  $c$  be an integer not equal to  $\pm 1$ . Let  $\theta_n(\chi_c)$  denote  $\chi$ -component of  $(1 - c\sigma_c^{-1})\theta_n$ :

$$\theta_n(\chi_c) = \sum_{a=0}^{p^n-1} \left( \frac{-1}{p^{n+1}} \sum_{\zeta \in W} r_n(\zeta \kappa^a) \chi(\zeta)^{-1} \right) \gamma_n^{-a}$$

where for every  $x \in \mathbb{Z}_p$

$$r_n(x) = \frac{s_n(x) - s_n(xc^{-1})c}{p^{n+1}} \in \mathbb{Z}.$$



Using the Fourier transform and the fact that under this map  $\gamma_n^{-a} \mapsto T^a \bmod 1 - T^{p^n}$ , we get

$$\begin{aligned} F_{\theta}^{\chi^c}(\kappa^{m-1}) &\equiv - \sum_{a=0}^{p^n-1} \sum_{\zeta \in W} r_n(\zeta \kappa^a) \chi(\zeta)^{-1} \kappa^{a(m-1)} \bmod p^{n+1} \\ &\equiv - \sum_{\substack{b=1 \\ (b,p)=1}}^{p^{n+1}} b^{m-1} r_n(b) \cdot \chi^* \omega^{-m}(b) \end{aligned}$$

where  $\chi \chi^* = \omega$ . The last line follows from the fact that  $\zeta \kappa^{a(m-1)} = \omega^{-m+1}(\zeta)(\zeta \kappa^a)^{m-1}$ . Notice the change in parity;  $\chi^*$  is an even character. The idea here is to try to take advantage of the following fact

$$\lim_{n \rightarrow \infty} \frac{1}{p^{n+1}} \sum_{\substack{b=1 \\ (b,p)=1}}^{p^{n+1}} b^m \cdot \chi^* \omega^{-m}(b) = (1 - \chi^* \omega^{-m}(p) p^{m-1}) B_{m, \chi^* \omega^{-m}}.$$

We use the congruence

$$b^{m-1} r_n(b) \cdot \chi^* \omega^{-m}(b) \equiv \frac{1}{mp^{n+1}} (s_n(b)^m - s_n(bc^{-1})^m c^m) \bmod p^{n+1}$$

to obtain

$$F_{\theta}^{\chi^c}(\kappa^{m-1}) \equiv \frac{-(1 - c^m \chi^* \omega^{-m}(c))}{m} \cdot \frac{1}{p^{n+1}} \sum_{\substack{b=1 \\ (b,p)=1}}^{p^{n+1}} b^m \cdot \chi^* \omega^{-m}(b) \bmod p^{n+1}.$$

Taking limits we get

$$\frac{F_{\theta}^{\chi^c}(\kappa^{m-1})}{1 - c^m \cdot \chi^* \omega^{-m}(c)} = L_p(1 - m, \chi^*).$$

Choose  $c$  so that  $\langle c \rangle = \kappa$ . Then we have

**Theorem 3.4** (Iwasawa [4]). *Let  $L_{\theta-}^{\chi}(T)$  be the function defined by*

$$L_{\theta-}^{\chi}(T) = \frac{F_{\theta}^{\chi^c}(T)}{1 - \kappa c \chi^{-1}(c) \cdot T}.$$

*Then  $L_{\theta-}^{\chi}(\kappa^s) = L_p(-s, \chi^*)$ .*

*Remark 3.5.* So the  $p$ -adic  $L$ -function attached to  $\chi^*$  (an even character) arises from the inverse limit of the  $\chi$ -parts (an odd character) of the classical Stickelberger elements  $\theta_n$ . Note that we can choose  $c$  so that  $1 - \kappa c \chi^{-1}(c) \cdot T$  is invertible (in  $\mathbb{Z}_p[[T - 1]]$ ) if and only if  $\chi \neq \omega$ . So if  $\chi \neq \omega$  (i.e.,  $\chi^* \neq 1$ ), we have  $L_{\theta-}^{\chi}(T) \in \mathbb{Z}_p[[T - 1]]$  and the isomorphisms

$$(3.1) \quad \frac{\mathbb{Z}_p(\Gamma)}{(\mu_{\theta}^{\chi})} \simeq \frac{\mathbb{Z}_p[[\Gamma]]}{(\theta(\chi))} \simeq \frac{\mathbb{Z}_p[[T - 1]]}{(L_{\theta-}^{\chi}(T))}.$$

This is a very satisfying result. Given Equations (1.1) and (1.4) and a host of other instances where  $L$ -functions appear to comment on algebraic structure, one wonders whether it's all just a cosmic coincidence or if it's a symptom of a deeper connection. This theorem points us towards the latter. In lieu of Theorem 3.1, one might even dare to hope that  $L_{\theta-}^{\chi}(T)$  is essentially the characteristic of the  $\mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T-1]]$ -module  $e_{\chi} \varprojlim A_n$  where  $A_n$  denotes the  $p$ -part of the class group of  $k_n$ . Miraculously, this is all true and proven by Mazur and Wiles [7].

Since  $L_{\theta-}^{\chi}(T)$  annihilates  $e_{\chi} \varprojlim A_n$ , it follows that the twisted power series  $L_{\theta-}^{\chi}(\frac{1}{\kappa T})$  annihilates  $e_{\chi^*} \varprojlim A_n^+$  where  $A_n^+$  denotes the  $p$ -part of the class group of  $k_n^+$ . Here's why: consider the  $\chi^*$ -part of  $\mathcal{X}_{\infty}$ , the Galois group of the maximal abelian pro- $p$  extension of  $\mathbb{Q}(\zeta_{p^{\infty}})^+$  unramified outside  $p$ . Note that  $e_{\chi^*} \mathcal{X}_{\infty}$  contains  $e_{\chi^*} \varprojlim A_n^+$  as a quotient, what's more,  $e_{\chi^*} \mathcal{X}_{\infty}$  is isomorphic to the Kummer dual of the  $\chi$ -part of  $\varinjlim A_n$ . But this latter module is essentially a twisted version of  $e_{\chi} \varprojlim A_n$  from which we may derive the claimed annihilation.

In the next section, we construct *explicit* annihilators of  $A_n^+$  for every level  $n \geq 0$  in the vein of the classical Stickelberger theorem. We begin with an explicit element that could be considered a “real” Gauss sum, and we explain how the factorization of this element gives rise to annihilators of  $A_n^+$ . These are new and build upon our work in [1]. We then show that these annihilators can be induced to give rise to the aforementioned twisted  $p$ -adic  $L$ -functions in a manner that parallels Iwasawa's work. This is interesting in its own right, but also indicates that these annihilators are, in a sense, full bodied.

#### 4. THE PLUS SIDE

Let  $k_n^+$  denote the maximal real subfield of  $k_n$ , and associate  $\Gamma_n$  with  $\text{Gal}(k_n^+/k_0^+)$ . For every positive integer  $n$ , we let

$$\delta_n(T) = \frac{\zeta_{p^{n+1}}^{g\kappa} - T}{\zeta_{p^{n+1}} - T}$$

where  $g$  is a primitive root mod  $p$ . We write  $\delta_n$  for  $\delta_n(1)$ ,  $K_n$  for  $\mathbb{Q}_p(\zeta_{p^{n+1}})$ , and  $\mathfrak{o}_n$  for  $\mathbb{Z}_p[\zeta_{p^{n+1}}]$ , the valuation integers of  $K_n$ .

**4.1. Stickelberger Elements.** Let  $\ell$  be a rational prime congruent to 1 mod  $p^{n+1}$ , and let  $\tau$  be a generator for  $\text{Gal}(k_n^+(\zeta_{\ell})/k_n^+)$ . Consider the “Gauss sum”

$$g_n^+(\ell) = - \sum_{b=1}^{\ell-1} \delta_n(\zeta_{\ell})^{N_b} \zeta_{\ell}^{\tau^b}$$

where

$$N_b = 1 + \tau + \cdots + \tau^{b-1} \in \mathbb{Z}[\text{Gal}(k_n^+(\zeta_{\ell})/k_n^+)].$$

The following lemma explains why we call this a Gauss sum.

**Lemma 4.1.** *The element  $g_n^+(\ell)$  is non-zero for some choice of  $\zeta_\ell$ . What's more, we have  $\delta_n(\zeta_\ell) \cdot g_n^+(\ell)^\tau = g_n^+(\ell)$ .*

*Proof.* Let  $\zeta_\ell = e^{2\pi i/\ell}$ . Let  $B$  be the  $(\ell - 1) \times (\ell - 1)$ -matrix

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \zeta_\ell^\tau & \zeta_\ell^{\tau^2} & \cdots & \zeta_\ell^{\tau^{\ell-2}} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_\ell^{\ell-2} & \zeta_\ell^{(\ell-2)\tau} & \cdots & \zeta_\ell^{(\ell-2)\tau^{\ell-2}} \end{bmatrix}$$

Now let  $A$  be the matrix  $B$  with the first row removed. Since  $\det B \neq 0$ , it follows that  $\dim \ker A = 1$ . In fact,  $\ker A$  is the span of the column vector

$$\begin{bmatrix} 1 - \zeta_\ell \\ \vdots \\ 1 - \zeta_\ell^{\tau^{\ell-2}} \end{bmatrix}, \text{ so } A \cdot \begin{bmatrix} \delta_n(\zeta_\ell)^{N_1} \\ \vdots \\ \delta_n(\zeta_\ell)^{N_{\ell-1}} \end{bmatrix} = \vec{0} \text{ implies that } \delta_n(\zeta_\ell) = \frac{1 - \zeta_\ell^\tau}{1 - \zeta_\ell}.$$

Now, using the fact that  $N_b \cdot \tau = N_{b+1} - 1$ , we see that

$$g_n^+(\ell)^\tau = -\delta_n(\zeta_\ell)^{-1} \sum_{c=2}^{\ell} \delta_n(\zeta_\ell)^{N_c} \zeta_\ell^{\tau^c}.$$

Since  $N_\ell = N + N_1$  where  $N$  is the norm, and  $\delta_n(\zeta_\ell)^N = 1$  (see [1, Theorem 3.1]), our lemma follows.  $\square$

*Remark 4.2.* Note that if one replaces  $k_n^+$  with  $k_n$  and  $\delta_n(\zeta_\ell)$  with  $\zeta_{p^{n+1}}$ , then we obtain the classical Gauss sum  $g_n(\ell)$ .

The fact that  $(g_n^+(\ell))$  is what Hilbert would call an “ambiguous” ideal is what enables its factorization to illuminate ideal relations; much like the classical Gauss sum  $g_n(\ell)$ . Unlike the classical case, the part of the factorization of  $g_n^+(\ell)$  that depends on  $\ell$  is much more difficult to isolate.

Toward that end, let  $\alpha : E_n^+ \rightarrow \mathbb{Z}_p[\text{Gal}(k_n^+/\mathbb{Q})]$  be a Galois module map where  $E_n^+$  denotes the units of  $k_n^+$  and fix an ideal class  $\mathfrak{c}$  in the Sylow  $p$ -subgroup of  $\mathfrak{C}(k_n^+)$ , the ideal class group. For every integer  $m > n$ , let  $\alpha_m(x)$  denote the group ring element of  $\mathbb{Z}[\text{Gal}(k_n^+/\mathbb{Q})]$  arrived at by applying the map  $s_m$  to the coefficients of  $\alpha(x)$ . Then there exists a rational integer  $\ell \equiv 1 \pmod{p^m}$  and a prime ideal  $\lambda \mid \ell$  of  $k_n^+$ , such that

$$(N(g_n^+(\ell))^{\tilde{\alpha}}) = \lambda^{\alpha_m(\delta_n)} \mathfrak{b}^{\ell-1}$$

where  $\mathfrak{b}$  is some integral ideal,  $\lambda \in \mathfrak{c}$ , and  $\tilde{\alpha}$  is, for lack of a better word, a “fudge-factor”. It follows that  $\alpha(\delta_n)$  annihilates  $\mathfrak{C}(k_n^+) \otimes \mathbb{Z}_p$ . In fact, it's shown in [1, Theorem 3.8] that if  $\alpha : E_n^+ \rightarrow \mathfrak{o}_n[\text{Gal}(k_n^+)/\mathbb{Q}]$  is a Galois module map, then  $\alpha(\delta_n)$  annihilates  $\mathfrak{C}(k_n^+) \otimes \mathfrak{o}_n$ . These ideas find their beginnings in a paper of Thaine [11] (see also [8]).

So all we need for a Stickelberger-esque theorem in this setting is an explicit map  $\alpha$  to complete the recipe. Consider the map

$$\alpha : k_n^\times \rightarrow K_n[\text{Gal}(k_n/\mathbb{Q})]$$

$$x \mapsto \sum_{\substack{a=1 \\ (a,p)=1}}^{p^{n+1}} \log_p(x^{\sigma_a}) \sigma_a^{-1},$$

where  $\log_p$  is the Iwasawa logarithm. Let  $W_n \subseteq E_n$  denote the roots of unity of  $k_n$  contained in the units of  $k_n$ , respectively. Since  $E_n = W_n E_n^+$ , it follows that  $\alpha(E_n) = \alpha(E_n^+)$ . We let  $\mathfrak{o}_n[\text{Gal}(k_n/\mathbb{Q})]$  act on any  $\mathfrak{o}_n[\text{Gal}(k_n^+/\mathbb{Q})]$ -module by restriction. Let  $\Theta_n = \alpha(\delta_n)$ . The element  $\Theta_n$  is a handsome analog of the classical Stickelberger element:

**Theorem 4.3** ([1, Theorem 1.1]). *For any  $\beta \in K_n[\text{Gal}(k_n/\mathbb{Q})]$  such that  $\beta \cdot \alpha(E_n) = \beta \cdot \alpha(E_n^+)$  is contained in  $\mathfrak{o}_n[\text{Gal}(k_n/\mathbb{Q})]$ , we have that  $\beta \Theta_n$  annihilates  $\mathfrak{C}(k_n^+) \otimes \mathfrak{o}_n$ .*

**4.2.  $p$ -adic  $L$ -functions.** Now, let  $\chi \neq 1$ . Let  $\Theta_n(\chi) \in K_n[\Gamma_n]$  denote the  $\chi$ -component of the real Stickelberger element  $\Theta_n$ . Note that

$$\Theta_n|_{k_n^+} = \sum_{a=0}^{p^n-1} \sum_{\zeta \in W/\pm 1} \log_p(\delta_n^{\sigma_{s_n(\zeta \kappa^a)} + \sigma_{s_n(-\zeta \kappa^a)}}) \sigma_{s_n(\zeta)}^{-1} \gamma_n^{-a}.$$

So if  $\chi$  is odd, then  $\Theta_n(\chi) = 0$ . So we can only get interesting results if  $\chi$  is an *even* character in this setting. Therefore we assume  $\chi$  is even for the remainder of this section. In this case, we have

$$\Theta_n(\chi) = (\chi(g)\gamma_n - 1) \sum_{a=0}^{p^n-1} \left( \sum_{\zeta \in W} \log_p(1 - \zeta_{p^{n+1}}^{\zeta \kappa^a}) \chi^{-1}(\zeta) \right) \gamma_n^{-a}.$$

**Lemma 4.4.** *The elements  $\Theta_n(\chi)$  are coherent, i.e., the sequence  $(\Theta_n(\chi))$  belongs to  $\mathbb{C}_p[[\Gamma]]$ .*

*Proof.* The elements  $\Theta_n(\chi)$  are coherent if and only if the map

$$M(a + p^n \mathbb{Z}_p) = \sum_{\zeta \in W} \log_p(1 - \zeta_{p^{n+1}}^{\zeta \kappa^a}) \chi^{-1}(\zeta)$$

forms a distribution since the sequence  $(\chi(g)\gamma_n - 1)$  is coherent. Since  $\kappa^{a+ip^n} \equiv \kappa^i(1 + ip^{n+1}) \pmod{p^{n+2}}$ , we have

$$\begin{aligned} \sum_{i=0}^{p-1} M(a + ip^n + p^{n+1} \mathbb{Z}_p) &= \sum_{\zeta \in W} \chi^{-1}(\zeta) \sum_{i=0}^{p-1} \log_p(1 - \zeta_{p^{n+2}}^{\chi(\zeta) \kappa^{a+ip^n}}) \\ &= \sum_{\zeta \in W} \chi^{-1}(\zeta) \log_p \prod_{i=0}^{p-1} (1 - \zeta_{p^{n+2}}^{\chi(\zeta) \kappa^a} \zeta_p^{i\chi(\zeta)}). \end{aligned}$$

Applying the relation

$$(4.1) \quad \prod_{i=0}^{p-1} (x - y\zeta_p^i) = x^p - y^p,$$

we get that  $M$  is a distribution, as claimed.  $\square$

Now,  $\Theta_n(\chi)$  is a coherent sequence and consequently we have a distribution  $\mu_\Theta^\chi$ , but as in the classical setting,  $\mu_\Theta^\chi$  is an unbounded distribution. What we need now is a collection of elements  $\beta_n(\chi) \in \mathfrak{o}_n[\Gamma_n]$  that *integralize* the elements  $\Theta_n(\chi)$ ; something akin to  $1 - c\sigma_c^{-1}$ . For a fixed  $\eta \in W$ , consider

$$v_n(\chi) = \sum_{a=0}^{p^n-1} \left( \sum_{\zeta \in W} \log_p (\eta - \zeta_{p^{n+1}}^{\zeta \kappa^a}) \chi^{-1}(\zeta) \right) \gamma_n^{-a}.$$

Since  $\eta^p = \eta$ , the same proof that shows  $\Theta_n(\chi)$  are coherent also shows that  $v_n(\chi)$  are coherent. The element  $v_n(\chi)$  has an inverse in  $K_n[\Gamma_n]$ , say  $\beta_n(\chi)$ , if and only if the  $\psi$ -part of  $v_n(\chi)$  is non-zero for every  $\psi \in \widehat{\Gamma}_n$ . Specifically, let  $e_\psi$  denote the idempotent

$$e_\psi = \frac{1}{p^n} \sum_{a=0}^{p^n-1} \psi(\kappa^a) \gamma_n^{-a}.$$

Let  $v_n(\chi\psi) \in K_n$  such that  $e_\psi v_n(\chi) = v_n(\chi\psi) e_\psi$ . We have

$$v_n(\chi\psi) = \sum_{a=0}^{p^n-1} \mu_v^\chi(a + p^n \mathbb{Z}_p) \psi^{-1}(\kappa^a) = \int_{\mathbb{Z}_p} \zeta_\psi^x d\mu_v^\chi(x),$$

where  $\zeta_\psi = \psi^{-1}(\kappa)$ . So if for all  $\psi \in \widehat{\Gamma}_n$  we have  $v(\chi\psi) \neq 0$ , then  $\beta_n(\chi)$  exists. In particular, we have

$$\beta_n(\chi) = \sum_{\psi \in \widehat{\Gamma}_n} \frac{e_\psi}{v_n(\chi\psi)} = \sum_{a=0}^{p^n-1} \left( \frac{1}{p^n} \sum_{\psi \in \widehat{\Gamma}_n} \frac{\psi(\kappa^a)}{\int_{\mathbb{Z}_p} \zeta_\psi^x d\mu_v^\chi(x)} \right) \gamma_n^{-a}.$$

Let  $\mu_\beta^\chi$  be the associated distribution. The convolution  $\mu_\beta^\chi * \mu_\Theta^\chi$  evaluated at  $a + p^n \mathbb{Z}_p$  is

$$\sum_{b=0}^{p^n-1} \mu_\Theta^\chi(a + p^n \mathbb{Z}_p) \left( \frac{1}{p^n} \sum_{\psi \in \widehat{\Gamma}_n} \frac{\zeta_\psi^{b-a}}{\int_{\mathbb{Z}_p} \zeta_\psi^x d\mu_v^\chi(x)} \right).$$

Recollecting terms we arrive at

$$(4.2) \quad \beta_n(\chi) \Theta_n(\chi) = \sum_{a=0}^{p^n-1} \left( \frac{1}{p^n} \sum_{\psi \in \widehat{\Gamma}_n} \zeta_\psi^{-a} \cdot \frac{\int_{\mathbb{Z}_p} \zeta_\psi^x d\mu_\Theta^\chi(x)}{\int_{\mathbb{Z}_p} \zeta_\psi^x d\mu_v^\chi(x)} \right) \gamma_n^{-a}.$$

We write  $\vartheta_n(\chi)$  for  $\beta_n(\chi) \Theta_n(\chi)$  and  $\mu_\vartheta^\chi$  for the associated distribution. For any choice of  $\eta$  (including  $\eta = 1$ ) and any choice of  $c \in \mathbb{Z}_p^\times$ , it's straightforward to verify that  $\mu_\vartheta^\chi$  is inert under  $\sigma_c$ . So  $\mu_\vartheta^\chi$  is  $\mathbb{Q}_p$ -valued. In fact,

as we shall see,  $\mu_\vartheta^\chi$  is  $\mathbb{Z}_p$ -valued so Equation (4.2) describes an explicit and coherent sequence of annihilators in  $\mathbb{Z}_p[[\Gamma]]$  of  $\mathfrak{C}(k_n^+) \otimes \mathbb{Z}_p$ .

**Proposition 4.5.** *For some choice of  $p-1$ -st root of unity  $\eta$ , the distribution  $\mu_\vartheta^\chi$  is  $\mathbb{Z}_p$ -valued.*

*Proof.* Let  $U_n$  denote the principal units of  $K_n$  and  $C_n$  the topological closure of the principal circular units of  $k_n$  in  $U_n$ . There is a choice of  $\eta$  such that

$$u_n = \left( \frac{\eta - \zeta_{p^{n+1}}}{\omega(\eta - 1)} \right)^{e_\chi}$$

is a  $\mathbb{Z}_p[\Gamma_n]$  generator for  $e_\chi U_n$  (see [3, 12]). What's more, the element

$$c_n = \left( \zeta_{p^{n+1}}^{(1-g\kappa)/2} \frac{\zeta_{p^{n+1}}^{g\kappa} - 1}{\zeta_{p^{n+1}} - 1} \right)^{(p-1) \cdot e_\chi}$$

is a  $\mathbb{Z}_p[\Gamma_n]$  generator for  $e_\chi C_n$  (see [9] for more on circular units). The definition for  $\alpha$  extends to a map  $A : K_n^\times \rightarrow K_n[\Gamma_n]$  defined in exactly the same way. Since  $A$  is a Galois map and  $\log_p$  vanishes at roots of unity, it follows that  $A(u_n) = v_n(\chi)$  and

$$A(c_n) = (p-1)\Theta_n(\chi).$$

Let  $\vartheta_n(\chi) \in \mathbb{Z}_p[\Gamma_n]$  such that  $u_n^{(p-1)\vartheta_n(\chi)} = c_n$ . Then

$$\vartheta_n(\chi)v_n(\chi) = \Theta_n(\chi).$$

Note that

$$\int_{\mathbb{Z}_p} \zeta_\psi^x d\mu_\Theta^\chi(x) = (\chi(g)\zeta_\psi^{-1} - 1) \cdot \sum_{\substack{a=1 \\ (a,p)=1}}^{p^{n+1}} \log_p(1 - \zeta_{p^{n+1}}^a) \chi\psi(a)^{-1} \neq 0$$

since  $\chi \neq 1$  and  $L_p(1, \chi\psi) \neq 0$ . So it must be that

$$\int_{\mathbb{Z}_p} \zeta_\psi^x d\mu_v^\chi(x) \neq 0,$$

Hence  $v_n(\chi)$  is invertible in  $K_n[\Gamma_n]$ . This gives us

$$\beta_n(\chi)\Theta_n(\chi) = \vartheta_n(\chi) \in \mathbb{Z}_p[\Gamma_n],$$

as desired. □

As noted before, this gives us the following

**Corollary 4.6.** *The sequence  $(\vartheta_n(\chi))$  in  $\mathbb{Z}_p[[\Gamma_n]]$  is such that  $\vartheta_n(\chi)$  annihilates  $e_\chi \cdot (\mathfrak{C}(k_n^+) \otimes \mathbb{Z}_p)$ .*

Now, let  $I_\vartheta^\chi$  denote the power series in  $\mathbb{Z}_p[[X]]$  obtained from  $(\vartheta_n(\chi))$  through the Iwasawa transform. Iwasawa [3] (see also [12, Theorem 13.56]) showed that there exists  $S(X) \in \mathbb{Z}_p[[X]]^\times$  such that

$$\frac{I_\vartheta^\chi(\kappa^s - 1)}{S(\kappa^s - 1)} = L_p(1 - s, \chi).$$

Let  $(\varsigma_n) \in \mathbb{Z}_p[[\Gamma]]$  such that  $I_\varsigma(X) = S(X)$ . Then

$$L_p(1-s, \chi) = \frac{I_\vartheta^\chi(X)}{I_\varsigma(X)} \Big|_{X=\kappa^s-1} = \frac{F_\vartheta^\chi(T)}{F_\varsigma(T)} \Big|_{T=\kappa^{-s}}$$

This gives us the following

**Theorem 4.7.** *Let  $L_{\theta^+}^\chi(T)$  be the function defined by*

$$L_{\theta^+}^\chi(T) = \frac{F_\vartheta^\chi(T)}{F_\varsigma(T)}.$$

Then  $L_{\theta^+}^\chi(\kappa^{-s}) = L_p(1-s, \chi)$ .

*Remark 4.8.* So the  $p$ -adic  $L$ -function attached to  $\chi$  (an even character) arises from the inverse limit of the  $\chi$ -parts of the real Stickelberger elements  $\vartheta_n$ . As before, for  $\chi \neq 1$ , we have  $L_{\theta^+}^\chi(T) \in \mathbb{Z}_p[[T-1]]$  and the isomorphisms

$$(4.3) \quad \frac{\mathbb{Z}_p(\Gamma)}{(\mu_\vartheta^\chi)} \simeq \frac{\mathbb{Z}_p[[\Gamma]]}{(\vartheta_n(\chi))} \simeq \frac{\mathbb{Z}_p[[T-1]]}{(L_{\theta^+}^\chi(T))}.$$

## 5. COMPARISON

Let  $\chi$  be an odd character not equal to  $\omega$ , so  $\chi^* \neq 1$ . From Equations (3.1) and (4.3) we get the following theorem.

**Theorem 5.1.** *For all  $s \in \mathbb{Z}_p$ , we have*

$$L_{\theta^-}^\chi(\kappa^{-s}) = L_p(s, \chi^*) = L_{\theta^+}^{\chi^*}(\kappa^{s-1}).$$

Consider the map  $\iota : \mathbb{Z}_p[[T-1]] \rightarrow \mathbb{Z}_p[[T-1]]$  defined by

$$\iota : F(T) \mapsto F\left(\frac{1}{\kappa T}\right).$$

The map  $\iota$  is well-defined since  $\frac{1}{\kappa T} \in 1 + (p, T-1)$ , what's more,  $\iota$  is an involution. Hence,  $\iota$  is an automorphism of  $\mathbb{Z}_p[[T-1]]$ . This gives us the following

**Corollary 5.2.** *We have the following equality of power series:*

$$L_{\theta^-}^\chi\left(\frac{1}{\kappa T}\right) = L_{\theta^+}^{\chi^*}(T) \in \mathbb{Z}_p[[T-1]].$$

*Proof.* This follows immediately from the fact that  $L_{\theta^-}^\chi(1/(\kappa T)) - L_{\theta^+}^{\chi^*}(T)$  vanishes on a neighborhood of 1 and the Weierstrass preparation theorem in  $\mathbb{Z}_p[[T-1]]$ .  $\square$

The elements  $\theta_n^+(\chi^*)$ , the sequence of group ring elements corresponding to  $L_{\theta^+}^{\chi^*}(T)$ , are of interest for the following reason.

**Proposition 5.3.** *For every non-negative integer  $n$ , the elements  $\theta_n^+(\chi^*)$  annihilate  $e_{\chi^*}(\mathfrak{C}(k_n^+) \otimes \mathbb{Z}_p)$ .*

*Proof.* From Corollary 4.6, we know that  $\vartheta_n(\chi^*)$  annihilates  $e_{\chi^*}(\mathfrak{C}(k_n^+) \otimes \mathbb{Z}_p)$ . Since  $F_\zeta(T)$  is an invertible power series in  $\mathbb{Z}_p[[T-1]]$ , it follows that  $\varsigma = (\varsigma_n)$  is invertible in  $\mathbb{Z}_p[[\Gamma]]$ . So  $\varsigma_n \in \mathbb{Z}_p[\Gamma_n]$  is invertible. It follows that if  $\vartheta_n(\chi^*)$  annihilates  $e_{\chi^*}(\mathfrak{C}(k_n^+) \otimes \mathbb{Z}_p)$ , then so must  $\vartheta_n(\chi^*) \cdot \varsigma_n^{-1} = \theta_n^+(\chi^*)$ .  $\square$

Using Theorem 5.1, we can derive an explicit expression for  $\mu_{\theta^+}^{\chi^*}$  which, in turn, gives an explicit expression for  $\vartheta_n^+(\chi)$ . Before giving this formula, we describe a little notation. Let  $\mathbf{1}_{a+p^n\mathbb{Z}_p}(x)$  denote the indicator function for  $a + p^n\mathbb{Z}_p$ ;  $\mathbf{1}_{a+p^n\mathbb{Z}_p}(x) = 1$  if  $x \in a + p^n\mathbb{Z}_p$  and 0 otherwise. For any distribution  $\mu \in \mathbb{Z}_p(\Gamma)$ , and any continuous function  $f$ , we define

$$\int_{a+p^n\mathbb{Z}_p} f(x) \, d\mu(x) = \int_{\mathbb{Z}_p} \mathbf{1}_{a+p^n\mathbb{Z}_p}(x) f(x) \, d\mu(x).$$

For any constant  $b \in \mathbb{Z}_p$ , the distribution  $\mu \circ b$  is defined by

$$(\mu \circ b)(a + p^n\mathbb{Z}_p) = \mu(ab + p^n\mathbb{Z}_p).$$

We write  $d\mu(bx)$  for  $d(\mu \circ b)(x)$ .

**Theorem 5.4.** *The distribution  $\mu_{\theta^+}^{\chi^*}$  is given by*

$$\mu_{\theta^+}^{\chi^*}(a + p^n\mathbb{Z}_p) = \int_{a+p^n\mathbb{Z}_p} \kappa^x \, d\mu_{\theta^-}^{\chi}(-x).$$

*Proof.* We compute the Fourier transform of the measure given in the theorem. Note that

$$\begin{aligned} \int_{\mathbb{Z}_p} T^x \, d\mu_{\theta^+}^{\chi^*}(x) &= \lim_{n \rightarrow \infty} \sum_{a=0}^{p^n-1} T^a \int_{a+p^n\mathbb{Z}_p} \kappa^x \, d\mu_{\theta^-}^{\chi}(-x) \\ &= \lim_{n \rightarrow \infty} \sum_{a=0}^{p^n-1} (\kappa T)^a \int_{a+p^n\mathbb{Z}_p} \kappa^{x-a} \, d\mu_{\theta^-}^{\chi}(-x). \end{aligned}$$

Consider the integral in the expression above. Let  $b \in a + p^n\mathbb{Z}_p$  and write  $b = a + cp^n$  where  $c \in \mathbb{Z}_p$ . For any  $c$ , we have  $\kappa^{cp^n} = 1 + c_n p^{n+1}$  for some  $c_n \in \mathbb{Z}_p$ . It follows that

$$\int_{a+p^n\mathbb{Z}_p} \kappa^{x-a} \, d\mu_{\theta^-}^{\chi}(-x) = \mu_{\theta^-}^{\chi}(-a + p^n\mathbb{Z}_p) + y_n(a)p^{n+1}$$

for some  $y_n(a) \in \mathbb{Z}_p$ . Hence

$$\begin{aligned} \int_{\mathbb{Z}_p} T^x \, d\mu_{\theta^+}^{\chi^*} &= \lim_{n \rightarrow \infty} \sum_{a=0}^{p^n-1} (\kappa T)^a \mu_{\theta^-}^{\chi}(-a + p^n\mathbb{Z}_p) + p^{n+1}(\kappa T)^a y_n(a) \\ &= \lim_{n \rightarrow \infty} \sum_{a=0}^{p^n-1} (\kappa T)^{-a} \mu_{\theta^-}^{\chi}(a + p^n\mathbb{Z}_p) \\ &= L_{\theta^-}^{\chi} \left( \frac{1}{\kappa T} \right). \end{aligned}$$



The theorem now follows from Theorem 5.1.  $\square$

*Remark 5.5.* Written out explicitly we have

$$\mu_{\theta^+}^{\chi^*}(a + p^n \mathbb{Z}_p) = \lim_{m \rightarrow \infty} \sum_{c=0}^{p^{m-n}-1} \kappa^{a+cp^n} \left( \frac{-1}{p^{n+1}} \sum_{\zeta \in W} s_n(\zeta \kappa^{-a-cp^n}) \chi(\zeta)^{-1} \right).$$

We summarize what we've shown in the following theorem.

**Theorem 5.6.** *Let  $\chi \neq \omega$  be an odd character and  $\chi^* = \chi^{-1}\omega$ . Then*

$$e_{\chi}(\mathfrak{C}(k_n) \otimes \mathbb{Z}_p)^{\theta_n^-(\chi)} = 0$$

$$e_{\chi^*}(\mathfrak{C}(k_n^+) \otimes \mathbb{Z}_p)^{\theta_n^+(\chi^*)} = 0 = e_{\chi^*}(\mathfrak{C}(k_n^+) \otimes \mathbb{Z}_p)^{\vartheta_n(\chi^*)}$$

The annihilators above are explicitly described in the following table along with their isomorphic images in  $\mathbb{Z}_p[[T-1]]$  and  $\mathbb{Z}_p[[\Gamma]]$ :

$\mathbb{Z}_p[[T-1]]$	$\mathbb{Z}_p(\Gamma)$	$\mathbb{Z}_p[[\Gamma]]$
$L_{\theta^-}^{\chi}(T)$	$\underbrace{\frac{-1}{p^{n+1}} \sum_{\zeta \in W} s_n(\zeta \kappa^a) \chi(\zeta)^{-1}}_{=\mu_{\theta^-}^{\chi}(a+p^n \mathbb{Z}_p)}$	$\underbrace{\sum_{a=0}^{p^n-1} \mu_{\theta^-}^{\chi}(a+p^n \mathbb{Z}_p) \gamma_n^{-a}}_{=\theta_n^-(\chi)}$
$F_{\zeta}(T) L_{\theta^+}^{\chi^*}(T)$	$\underbrace{\frac{1}{p^n} \sum_{\psi \in \widehat{\Gamma}_n} \zeta_{\psi}^{-a} \cdot \frac{\int_{\mathbb{Z}_p} \zeta_{\psi}^x d\mu_{\Theta}^{\chi^*}(x)}{\int_{\mathbb{Z}_p} \zeta_{\psi}^x d\mu_{\psi}^{\chi^*}(x)}}_{=\mu_{\vartheta}^{\chi^*}(a+p^n \mathbb{Z}_p)}$	$\underbrace{\sum_{a=0}^{p^n-1} \mu_{\vartheta}^{\chi^*}(a+p^n \mathbb{Z}_p) \gamma_n^{-a}}_{=\vartheta_n(\chi^*)}$
$L_{\theta^+}^{\chi^*}(T)$	$\underbrace{\int_{a+p^n \mathbb{Z}_p} \kappa^x d\mu_{\theta^-}^{\chi}(-x)}_{=\mu_{\theta^+}^{\chi^*}(a+p^n \mathbb{Z}_p)}$	$\underbrace{\sum_{a=0}^{p^n-1} \mu_{\theta^+}^{\chi^*}(a+p^n \mathbb{Z}_p) \gamma_n^{-a}}_{=\theta_n^+(\chi^*)}$

Moreover, we have

$$L_{\theta^-}^{\chi}(\kappa^{-s}) = L_p(s, \chi^*) = L_{\theta^+}^{\chi^*}(\kappa^{s-1}).$$

The expressions for  $\theta_n^+(\chi^*)$  and  $\vartheta_n(\chi^*)$  are interesting as they may shed some new light on the quality of  $\mathfrak{C}(k_n^+) \otimes \mathbb{Z}_p$ . Can  $\theta_n^+(\chi^*)$  be made more explicit? And what about  $\vartheta_n(\chi^*)$ . In a recent paper [2], Waller and the author showed that although the distributions  $\mu_{\Theta}^{\chi^*}$  and  $\mu_{\psi}^{\chi^*}$  are unbounded in value they nonetheless form  $C^1(\mathbb{Z}_p)$  functionals through *Volkenborn* integration. In fact, the Fourier transform of  $\mu_{\beta}^{\chi^*}$  interpolates the kind of Gauss sums that appear in formulas for  $L_p(1, \chi^*)$ . Can they be used to shed light on the nature of  $\vartheta_n(\chi^*)$ ? We hope to address these questions in the future.

*Acknowledgements.* Many thanks to the anonymous referee for their careful review and for numerous suggestions that have improved the quality of this paper. I also want to thank David Goss for his encouragement during the writing of this manuscript.

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